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An equivalence theorem for a massive spin one particle interacting with Dirac particles in quantum field theory

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Abstract. The $(1,0) \oplus (0,1)$ massive spin one field $\phi_{\mu\nu}(x)$ is quantized following the method of Takahashi and Umezawa. The interaction Hamiltonian in the interaction picture is then calculated for such a field interacting in a simple way with Dirac fields. In contrast with a remark of Kyriakopoulos, the generalized Matthews' theorem is found to apply to the calculation of S matrix elements. The analogous theory in terms of the Proca field is given in outline, and the two theories are found to be inequivalent. A discussion of how the inequivalence arises is given; then it is demonstrated how equivalent theories may be constructed, and the equivalence is proved.

1. Introduction

In relativistic quantum field theory a massive spin one particle is more often than not described by the Proca field $V_\mu(x)$ which transforms under the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group $(\mathrm{SU}(2) \otimes \mathrm{SU}(2)$ decomposition). Such a mode of description is not however unique. In fact, as has been pointed out by Weinberg (1964a, 1964b, 1969), it is possible to describe a spin one particle by a field which transforms under any representation of the Lorentz group satisfying the condition that, when it is restricted to the rotation subgroup, the representation contains the spin one amongst its components. Thus another simple possibility is to describe a spin one particle by an antisymmetric tensor field $\phi_{\mu\nu}(x)$ which transforms under the $(1,0) \oplus (0,1)$ representation of the Lorentz group. The use of such a field is attractive for two reasons. Firstly, the $(1,0) \oplus (0,1)$ representation, when restricted to the rotation subgroup, unlike the $(\frac{1}{2}, \frac{1}{2})$ representation, only contains the spin one amongst its components. This has the advantage that one is not involved with the subsidiary conditions usually needed to remove unwanted spins from the field. Secondly, the antisymmetric tensor field is the direct generalization to spin one of the Dirac equation for spin half particles.

Now it is known that in the free-field case the antisymmetric tensor field and the Proca field are entirely equivalent (Kyriakopoulos 1969). However when interactions are introduced the situation is not so simple, and these two fields will in general give rise to inequivalent theories. This being said, it is the purpose of this paper to explore in detail, in the example of a spin one particle interacting with Dirac particles, how such inequivalent theories arise, how they differ and finally how their differences may be resolved.

The plan of the paper is as follows. In § 2 the Lagrangian for the system of the antisymmetric tensor field and the Dirac fields is set up, and the interaction Hamiltonian is calculated following the method of Takahashi and Umezawa (1953) (Takahashi 1969).

This process is then repeated in outline only in § 3 for the well known case of the Proca field interacting with the Dirac fields. The inequivalence of these two theories is commented on in § 4, and it is shown how the interactions must be modified in order that the differences may be resolved. § 5 is devoted to a discussion and conclusions.

2. The antisymmetric tensor field

For simplicity, the antisymmetric tensor field $\phi_{\mu\nu}(x)$ is taken to be hermitian, and its free Lagrangian is given by†

$$\mathscr{L}(x) = \tfrac{1}{2}\phi^{\mu\nu}(x)(A_{\mu\nu\lambda\rho}(\partial)+\mu^2 l_{\mu\nu\lambda\rho})\phi^{\lambda\rho}(x)$$

where

$$A_{\mu\nu\lambda\rho}(\partial) \equiv \tfrac{1}{2}(g_{\mu\lambda}\partial_\nu\partial_\rho+g_{\nu\rho}\partial_\mu\partial_\lambda-g_{\mu\rho}\partial_\nu\partial_\lambda-g_{\nu\lambda}\partial_\mu\partial_\rho) \tag{1}$$

and

$$l_{\mu\nu\lambda\rho} \equiv \tfrac{1}{2}(g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}). \tag{2}$$

Whence follows the equation of motion

$$\Lambda_{\mu\nu\lambda\rho}(\partial)\phi^{\lambda\rho}(x) \equiv (A_{\mu\nu\lambda\rho}(\partial)+\mu^2 l_{\mu\nu\lambda\rho})\phi^{\lambda\rho}(x) = 0 \tag{3}$$

the solutions of which to be considered here being restricted throughout to those antisymmetric in the tensor indices.

Incidentally two points concerning the equation (3) may be noted. Firstly, it is precisely the equation considered by Kyriakopoulos (1969); however it will be treated here in a manner which differs from his approach, some comparative remarks being made in § 5. Secondly, it is dual to the generalization of the Weinberg equation for spin one (Weinberg 1964a) considered by Hammer *et al* (1968) written in tensor form.

The field $\phi_{\mu\nu}(x)$ may now be quantized by the method of Takahashi and Umezawa (1953); and, following Takahashi (1969), the Klein–Gordon divisor, $d_{\mu\nu\lambda\rho}(\partial)$, defined by

$$\Lambda_{\mu\nu\lambda\rho}(\partial)d^{\lambda\rho}_{\alpha\beta}(\partial) = (\partial^2+\mu^2)l_{\mu\nu\alpha\beta}$$

is readily found to be

$$d_{\mu\nu\lambda\rho}(\partial) = -\frac{1}{\mu^2}(A_{\mu\nu\lambda\rho}(\partial)-(\partial^2+\mu^2)l_{\mu\nu\lambda\rho}). \tag{4}$$

The commutator of the field $\phi_{\mu\nu}(x)$ is now given in terms of the Klein–Gordon divisor as follows:

$$[\phi_{\mu\nu}(x), \phi_{\lambda\rho}(x')] = -\mathrm{i}d_{\mu\nu\lambda\rho}(\partial)\Delta(x-x') \tag{5}$$

and the free-particle propagator is given by

$$\begin{aligned}\langle T(\phi_{\mu\nu}(x), \phi_{\lambda\rho}(x'))\rangle_0 = &-\mathrm{i}d_{\mu\nu\lambda\rho}(\partial)\Delta_c(x-x')\\ &-\mathrm{i}[\theta(x_0-x_0'), d_{\mu\nu\lambda\rho}(\partial)]\Delta(x-x')\end{aligned} \tag{6}$$

† The metric used in this paper is $(g_{\mu\nu}) = \mathrm{diag}\,(1, -1, -1, -1)$.

where

$$\Delta_c(x) = \theta(x_0)\Delta^+(x) - \theta(-x_0)\Delta^-(x)$$

and $\Delta^\pm(x)$, $\Delta(x)$ are the well known solutions of the Klein–Gordon equation.

The position has now been reached where the interaction between the above field and Dirac fields may be considered, and, on the grounds of simplicity, the Lagrangian is taken to be†

$$\mathscr{L}(x) = \overline{\psi}(x)(\mathrm{i}\partial\!\!\!/ - m)\psi(x) + \tfrac{1}{2}\phi^{\mu\nu}(x)(A_{\mu\nu\lambda\rho}(\partial) + \mu^2 l_{\mu\nu\lambda\rho})\phi^{\lambda\rho}(x)$$
$$-\frac{\mu}{\sqrt{2}}J_{\mu\nu}(x)\phi^{\mu\nu}(x) - \frac{1}{\sqrt{2}\mu}(J_\mu(x)\partial_\nu - J_\nu(x)\partial_\mu)\phi^{\mu\nu}(x) \tag{7}$$

where

$$J_{\mu\nu}(x) = g_1\overline{\psi}(x)\sigma_{\mu\nu}\psi(x) \tag{8}$$

and

$$J_\mu(x) = g_2\overline{\psi}(x)\gamma_\mu\psi(x). \tag{9}$$

The consequent equations of motion are

$$(\mathrm{i}\partial\!\!\!/ - m)\psi(x) = \frac{\mu}{\sqrt{2}}g_1\sigma_{\mu\nu}\psi(x)\phi^{\mu\nu}(x) + \frac{g_2}{\sqrt{2}\,\mu}(\gamma_\mu\psi(x)\partial_\nu$$
$$-\gamma_\nu\psi(x)\partial_\mu)\phi^{\mu\nu}(x) \equiv I(x) \tag{10}$$

$$(A_{\mu\nu\lambda\rho}(\partial) + \mu^2 l_{\mu\nu\lambda\rho})\phi^{\lambda\rho}(x) = \frac{\mu}{\sqrt{2}}J_{\mu\nu}(x) + \frac{1}{\sqrt{2}\,\mu}(\partial_\mu J_\nu(x) - \partial_\nu J_\mu(x)). \tag{11}$$

These equations are now quantized, and the interaction Hamiltonian found following the method of Takahashi and Umezawa (1953) (Takahashi 1969).

Firstly equations (10) and (11) are solved by the method of Green functions to give

$$\psi(x) = Z_2^{1/2}\psi(x) - \int_{-\infty}^{\infty} d(\partial)\Delta^{\mathrm{ret}}(x-x')I(x')\,\mathrm{d}^4x' \tag{12}$$

$$\phi_{\mu\nu}(x) = Z_3^{1/2}\phi_{\mu\nu}(x) - \int_{-\infty}^{\infty} d_{\mu\nu\lambda\rho}(\partial)\Delta^{\mathrm{ret}}(x-x')\frac{\mu}{\sqrt{2}}J^{\lambda\rho}(x')\,\mathrm{d}^4x'$$
$$-\int_{-\infty}^{\infty}\partial^\lambda d_{\mu\nu\lambda\rho}(\partial)\Delta^{\mathrm{ret}}(x-x')\frac{\sqrt{2}}{\mu}J^\rho(x')\,\mathrm{d}^4x' \tag{13}$$

where

$$\Delta^{\mathrm{ret}}(x) = \theta(x_0)\Delta(x)$$

and

$$d(\partial) = -(\mathrm{i}\partial\!\!\!/ + m)$$

† From this stage on, quantities written as, for example $\phi_{\mu\nu}(x)$ and $\phi_{\mu\nu}(x)$ will be taken as Heisenberg and interaction picture quantities respectively.

is the Klein–Gordon divisor of the Dirac field. Next the auxiliary fields are defined by

$$\psi(x,\sigma) = Z_2^{1/2}\psi(x) - \int_{-\infty}^{\sigma} d(\partial)\Delta(x-x')\boldsymbol{I}(x')\,\mathrm{d}^4x' \tag{14}$$

$$\phi_{\mu\nu}(x,\sigma) = Z_3^{1/2}\phi_{\mu\nu}(x) - \int_{-\infty}^{\sigma} d_{\mu\nu\lambda\rho}(\partial)\Delta(x-x')\frac{\mu}{\sqrt{2}}\boldsymbol{J}^{\lambda\rho}(x')\,\mathrm{d}^4x' - \int_{-\infty}^{\sigma} \partial^{\lambda}d_{\mu\nu\lambda\rho}(\partial)\Delta(x-x')\frac{\sqrt{2}}{\mu}\boldsymbol{J}^{\rho}(x')\,\mathrm{d}^4x' \tag{15}$$

where the point x need not necessarily lie on the spacelike surface σ. It now follows from equations (12)–(15) that

$$\psi(x/\sigma) = \boldsymbol{\psi}(x) - \int_{-\infty}^{\infty} [\theta(x_0-x_0'), d(\partial)]\Delta(x-x')\boldsymbol{I}(x')\,\mathrm{d}^4x' \tag{16}$$

$$\phi_{\mu\nu}(x/\sigma) = \boldsymbol{\phi}_{\mu\nu}(x) - \int_{-\infty}^{\infty} [\theta(x_0-x_0'), d_{\mu\nu\lambda\rho}(\partial)]\Delta(x-x')\frac{\mu}{\sqrt{2}}\boldsymbol{J}^{\lambda\rho}(x')\,\mathrm{d}^4x' - \int_{-\infty}^{\infty} [\theta(x_0-x_0'), \partial^{\lambda}d_{\mu\nu\lambda\rho}(\partial)]\Delta(x-x')\frac{\sqrt{2}}{\mu}\boldsymbol{J}^{\rho}(x')\,\mathrm{d}^4x' \tag{17}$$

$$\partial^{\mu}\phi_{\mu\nu}(x/\sigma) = \partial^{\mu}\boldsymbol{\phi}_{\mu\nu}(x) - \int_{-\infty}^{\infty} [\theta(x_0-x_0'), \partial^{\mu}d_{\mu\nu\lambda\rho}(\partial)]\Delta(x-x')\frac{\mu}{\sqrt{2}}\boldsymbol{J}^{\lambda\rho}(x')\,\mathrm{d}^4x' - \int_{-\infty}^{\infty} [\theta(x_0-x_0'), \partial^{\mu}\partial^{\lambda}d_{\mu\nu\lambda\rho}(\partial)]\Delta(x-x')\frac{\sqrt{2}}{\mu}\boldsymbol{J}^{\rho}(x')\,\mathrm{d}^4x' \tag{18}$$

where now the notation x/σ means that the point x is restricted to being on the spacelike surface σ. On use of the identities

$$[\theta(x_0), \partial_{\mu}]\Delta(x) = 0 \tag{19}$$

$$[\theta(x_0), \partial_{\mu}\partial_{\nu}]\Delta(x) = \eta_{\mu}\eta_{\nu}\delta^{(4)}(x) \tag{20}$$

where η_{μ} is the unit normal to the spacelike surface σ, equations (16)–(18) reduce respectively to

$$\psi(x/\sigma) = \boldsymbol{\psi}(x) \tag{21}$$

$$\phi_{\mu\nu}(x/\sigma) = \boldsymbol{\phi}_{\mu\nu}(x) - \frac{1}{\sqrt{2}\,\mu}\boldsymbol{J}_{\mu\nu}(x) + \frac{1}{\sqrt{2}\,\mu}(\boldsymbol{J}_{\mu\rho}(x)\eta^{\rho}\eta_{\nu} - \boldsymbol{J}_{\nu\rho}(x)\eta^{\rho}\eta_{\mu}) \tag{22}$$

$$\partial^{\mu}\phi_{\mu\nu}(x/\sigma) = \partial^{\mu}\boldsymbol{\phi}_{\mu\nu}(x) - \frac{1}{\sqrt{2}\,\mu}\boldsymbol{J}_{\nu}(x) + \frac{1}{\sqrt{2}\,\mu}\boldsymbol{J}_{\rho}(x)\eta^{\rho}\eta_{\nu}. \tag{23}$$

The currents can now be expressed in terms of the auxiliary fields through equations (21)–(23) as follows:

$$\boldsymbol{J}_{\mu}(x) = J_{\mu}(x/\sigma) \tag{24}$$

$$\boldsymbol{J}_{\mu\nu}(x) = J_{\mu\nu}(x/\sigma) \tag{25}$$

$$\boldsymbol{I}(x) = \frac{\mu g_1}{\sqrt{2}}\sigma_{\mu\nu}\psi(x/\sigma)\left(\phi^{\mu\nu}(x/\sigma)+\frac{1}{\sqrt{2}\,\mu}J^{\mu\nu}(x/\sigma)-\frac{1}{\sqrt{2}\,\mu}(J^{\mu\rho}(x/\sigma)\eta_\rho\eta^\nu\right.$$
$$\left.-J^{\nu\rho}(x/\sigma)\eta_\rho\eta^\mu)\right)+\frac{g_2}{\sqrt{2}\,\mu}(\gamma_\mu\psi(x/\sigma)\partial_\nu-\gamma_\nu\psi(x/\sigma)\partial_\mu)\phi^{\mu\nu}(x/\sigma)$$
$$-\frac{g_2}{\mu^2}\gamma_\mu\psi(x/\sigma)J^\mu(x/\sigma)+\frac{g_2}{\mu^2}\gamma_\mu\psi(x/\sigma)J_\rho(x/\sigma)\eta^\mu\eta^\rho. \tag{26}$$

Finally, on noting that the auxiliary fields satisfy the same commutation relations as their Heisenberg picture counterparts, for example equation (5) for the antisymmetric tensor field, the interaction Hamiltonian density is calculated from the equations

$$[\psi(x,\sigma), \mathscr{H}_{\mathrm{int}}(\eta, x'/\sigma)] = -\mathrm{i}d(\partial)\Delta(x-x')\boldsymbol{I}(x')$$
$$[\phi_{\mu\nu}(x,\sigma), \mathscr{H}_{\mathrm{int}}(\eta, x'/\sigma)] = -\mathrm{i}d_{\mu\nu\lambda\rho}(\partial)\Delta(x-x')\frac{\mu}{\sqrt{2}}\boldsymbol{J}^{\lambda\rho}(x')$$
$$-\mathrm{i}\partial^\lambda d_{\mu\nu\lambda\rho}(\partial)\Delta(x-x')\frac{\sqrt{2}}{\mu}\boldsymbol{J}^\rho(x')$$

where the currents on the right hand side are to be taken as expressed in terms of the auxiliary fields through equations (24)–(26). Having calculated $\mathscr{H}_{\mathrm{int}}(\eta, x/\sigma)$ from the above two equations, the interaction Hamiltonian density is obtained from it merely by replacing the auxiliary fields therein by their interaction picture counterparts. Thus

$$\mathscr{H}_{\mathrm{int}}(\eta, x) = \frac{\mu}{\sqrt{2}}J_{\mu\nu}(x)\phi^{\mu\nu}(x)+\frac{1}{\sqrt{2}\,\mu}(J_\mu(x)\partial_\nu-J_\nu(x)\partial_\mu)\phi^{\mu\nu}(x)$$
$$+\frac{1}{4}J_{\mu\nu}(x)J^{\mu\nu}(x)-\frac{1}{2\mu^2}J_\mu(x)J^\mu(x)$$
$$-\frac{1}{2}J_{\mu\nu}(x)J^{\mu\rho}(x)\eta^\nu\eta_\rho+\frac{1}{2\mu^2}(J_\mu(x)\eta^\mu)^2. \tag{27}$$

It will now be shown that in calculating the S matrix to second order in the coupling constants the interaction Hamiltonian may be taken to be effectively

$$\mathscr{H}_{\mathrm{int}}(x) = \frac{\mu}{\sqrt{2}}J_{\mu\nu}(x)\phi^{\mu\nu}(x)+\frac{1}{\sqrt{2}\,\mu}(J_\mu(x)\partial_\nu-J_\nu(x)\partial_\mu)\phi^{\mu\nu}(x) \tag{28}$$

provided that, whenever a contraction of the antisymmetric tensor field occurs, the T^* product, defined by

$$\langle T^*(\phi_{\mu\nu}(x), \phi_{\lambda\rho}(x'))\rangle_0 = -\mathrm{i}d_{\mu\nu\lambda\rho}(\partial)\Delta_{\mathrm{c}}(x-x') \tag{29}$$

is used in place of the full expression for the free-particle propagator given by equation (6). The extension to all orders in the coupling constants follows exactly the method of Kyriakopoulos (1969), is tedious and is consequently omitted.

To second order in the coupling constants the S matrix is given by Dyson's formula as

$$S = 1 - \mathrm{i}\int_{-\infty}^{\infty} T(\mathscr{H}_{\mathrm{int}}(\eta, x))\,\mathrm{d}^4x + \frac{(-\mathrm{i})^2}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T(\mathscr{H}_{\mathrm{int}}(\eta, x)\mathscr{H}_{\mathrm{int}}(\eta, x'))$$
$$\times\, \mathrm{d}^4x\,\mathrm{d}^4x' + \ldots$$
$$= \mathrm{i}\int_{-\infty}^{\infty} T\bigg(\frac{1}{2\mu^2}J_\mu(x)J^\mu(x) - \frac{1}{4}J_{\mu\nu}(x)J^{\mu\nu}(x) - \frac{1}{2\mu^2}(J_\mu(x)\eta^\mu)^2$$
$$+\frac{1}{2}J_{\mu\nu}(x)J^{\mu\rho}(x)\eta^\nu\eta_\rho\bigg)\,\mathrm{d}^4x - \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T\bigg(\frac{\mu^2}{2}J_{\mu\nu}(x)J_{\lambda\rho}(x')$$
$$\times\langle T(\phi^{\mu\nu}(x), \phi^{\lambda\rho}(x'))\rangle_0 + \frac{2}{\mu^2}J_\mu(x)J_\nu(x')\langle T(\partial_\lambda\phi^{\mu\lambda}(x), \partial'_\rho\phi^{\nu\rho}(x'))\rangle_0$$
$$+J_{\mu\nu}(x)J_\lambda(x')\langle T(\phi^{\mu\nu}(x), \partial'_\rho\phi^{\lambda\rho}(x'))\rangle_0$$
$$+J_\mu(x)J_{\lambda\rho}(x')\langle T(\partial_\nu\phi^{\mu\nu}(x), \phi^{\lambda\rho}(x'))\rangle_0\bigg)\,\mathrm{d}^4x\,\mathrm{d}^4x'$$

+ terms not of second order in the coupling constants and terms involving only external spin one particles. (30)

The expression (6), together with the identities (19) and (20) are used to give the following expressions for the propagators appearing in the above:

$$\langle T(\phi_{\mu\nu}(x), \phi_{\lambda\rho}(x'))\rangle_0 = -\mathrm{i}d_{\mu\nu\lambda\rho}(\partial)\Delta_{\mathrm{c}}(x-x') + \frac{\mathrm{i}}{\mu^2}(A_{\mu\nu\lambda\rho}(\eta) - l_{\mu\nu\lambda\rho})\delta^{(4)}(x-x')$$

$$\langle T(\phi_{\mu\nu}(x), \partial'^\lambda\phi_{\lambda\rho}(x'))\rangle_0 = -\mathrm{i}\partial'^\lambda d_{\mu\nu\lambda\rho}(\partial)\Delta_{\mathrm{c}}(x-x')$$
$$\langle T(\partial^\mu\phi_{\mu\nu}(x), \phi_{\lambda\rho}(x'))\rangle_0 = -\mathrm{i}\partial^\mu d_{\mu\nu\lambda\rho}(\partial)\Delta_{\mathrm{c}}(x-x')$$
$$\langle T(\partial^\mu\phi_{\mu\nu}(x), \partial'^\lambda\phi_{\lambda\rho}(x'))\rangle_0 = -\mathrm{i}\partial^\mu\partial'^\lambda d_{\mu\nu\lambda\rho}(\partial)\Delta_{\mathrm{c}}(x-x')$$
$$+\mathrm{i}l_{\mu\nu\lambda\rho}\eta^\mu\eta^\lambda\delta^{(4)}(x-x').$$

Now it is clear by inspection that a substitution of these expressions into (30) leads to the desired cancellations, and thus the prescription given by equations (28) and (29) may be used for the calculation of the S matrix.

It should be noted that this result contrasts with the results of Kyriakopoulos (1969) and a discussion of this point is given in § 5.

3. The Proca field

The Proca field is here assumed to be hermitian, and given by the free Lagrangian

$$\mathscr{L}(x) = -\tfrac{1}{4}(\partial_\mu V_\nu(x) - \partial_\nu V_\mu(x))(\partial^\mu V^\nu(x) - \partial^\nu V^\mu(x)) + \tfrac{1}{2}\mu^2 V_\mu(x)V^\mu(x)$$

whence the equation of motion

$$-((\partial^2+\mu^2)g_{\mu\nu} - \partial_\mu\partial_\nu)V^\nu(x) \equiv \Lambda_{\mu\nu}(\partial)V^\nu(x) = 0. \tag{31}$$

This field may again be quantized following the method of Takahashi and Umezawa, and the Klein–Gordon divisor, defined by

$$\Lambda_{\mu\nu}(\partial)d^{\nu}{}_{\lambda}(\partial) = (\partial^2+\mu^2)g_{\mu\lambda}$$

is easily calculated to be

$$d_{\mu\nu}(\partial) = g_{\mu\nu}+\frac{\partial_\mu\partial_\nu}{\mu^2}. \tag{32}$$

Thus the commutator of the field $V_\mu(x)$ is

$$[V_\mu(x),\ V_\nu(x')] = -\mathrm{i}d_{\mu\nu}(\partial)\Delta(x-x') \tag{33}$$

and the free-particle propagator is given by

$$\langle T(V_\mu(x),\ V_\nu(x'))\rangle_0 = -\mathrm{i}d_{\mu\nu}(\partial)\Delta_c(x-x')-\mathrm{i}[\theta(x_0-x_0'),\ d_{\mu\nu}(\partial)]\Delta(x-x'). \tag{34}$$

The interaction between the Proca field and the Dirac fields is, again on grounds of simplicity, introduced by means of the Lagrangian

$$\begin{aligned}\mathscr{L}(x) = {}&\bar{\psi}(x)(\mathrm{i}\partial\!\!\!/-m)\psi(x)-\tfrac{1}{4}(\partial^\mu V^\nu(x)-\partial^\nu V^\mu(x))(\partial_\mu V_\nu(x)-\partial_\nu V_\mu(x))\\ &+\tfrac{1}{2}\mu^2 V_\mu(x)V^\mu(x)-\tfrac{1}{2}\boldsymbol{J}_{\mu\nu}(x)(\partial^\mu V^\nu(x)-\partial^\nu V^\mu(x))-\boldsymbol{J}_\mu(x)V^\mu(x)\end{aligned} \tag{35}$$

where the currents $\boldsymbol{J}_\mu(x)$ and $\boldsymbol{J}_{\mu\nu}(x)$ are again given by equations (8) and (9). The equations of motion following from (35) are

$$(\mathrm{i}\partial\!\!\!/-m)\psi(x) = \frac{g_1}{2}\sigma_{\mu\nu}\psi(x)(\partial^\mu V^\nu(x)-\partial^\nu V^\mu(x))+g_2\gamma_\mu\psi(x)V^\mu(x) \tag{36}$$

$$((\partial^2+\mu^2)g_{\mu\nu}-\partial_\mu\partial_\nu)V^\nu(x) = \boldsymbol{J}_\mu(x)-\partial^\nu\boldsymbol{J}_{\nu\mu}(x). \tag{37}$$

Starting from these equations the method of Takahashi and Umezawa is again used to calculate the interaction Hamiltonian. Since the calculation is completely analogous to that of § 2 the details are omitted, and the final form of the interaction Hamiltonian is merely quoted, namely

$$\begin{aligned}\mathscr{H}_{\mathrm{int}}(\eta,x) = {}&\tfrac{1}{2}J_{\mu\nu}(x)(\partial^\mu V^\nu(x)-\partial^\nu V^\mu(x))+J_\mu(x)V^\mu(x)\\ &-\frac{1}{2}J_{\mu\nu}(x)J^{\mu\lambda}(x)\eta^\nu\eta_\lambda+\frac{1}{2\mu^2}(J_\mu(x)\eta^\mu)^2.\end{aligned} \tag{38}$$

It will now be shown that in calculating the S matrix to second order in the coupling constants the interaction Hamiltonian may be taken to be effectively

$$\mathscr{H}_{\mathrm{int}}(x) = \tfrac{1}{2}J_{\mu\nu}(x)(\partial^\mu V^\nu(x)-\partial^\nu V^\mu(x))+J_\mu(x)V^\mu(x) \tag{39}$$

provided that, whenever a contraction of the vector field occurs, the T^* product, defined by

$$\langle T^*(V_\mu(x),\ V_\nu(x'))\rangle_0 = -\mathrm{i}d_{\mu\nu}(\partial)\Delta_c(x-x') \tag{40}$$

is used in place of the full expression for the free-particle propagator given by equation (34). The extension to all orders again follows the argument of Kyriakopoulos (1969) and is omitted.

On including only second order terms which do not involve only external vector fields, the S matrix is given by Dyson's formula as

$$S = \mathrm{i}\int_{-\infty}^{\infty} T\left(\frac{1}{2}J_{\mu\nu}(x)J^{\mu\lambda}(x)\eta^{\nu}\eta_{\lambda} - \frac{1}{2\mu^2}(J_{\mu}(x)\eta^{\mu})^2\right)\mathrm{d}^4x$$

$$-\tfrac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T\{\tfrac{1}{4}J_{\mu\nu}(x)J_{\lambda\rho}(x')\langle T(\partial^{\mu}V^{\nu}(x)-\partial^{\nu}V^{\mu}(x), \partial'^{\lambda}V^{\rho}(x')$$

$$-\partial'^{\rho}V^{\lambda}(x'))\rangle_0 + J_{\mu}(x)J_{\nu}(x')\langle T(V^{\mu}(x), V^{\nu}(x'))\rangle_0$$

$$+\tfrac{1}{2}J_{\mu\nu}(x)J_{\lambda}(x')\langle T(\partial^{\mu}V^{\nu}(x)-\partial^{\nu}V^{\mu}(x), V^{\lambda}(x'))\rangle_0$$

$$+\tfrac{1}{2}J_{\mu}(x)J_{\lambda\rho}(x')\langle T(V^{\mu}(x), \partial'^{\lambda}V^{\rho}(x')-\partial'^{\rho}V^{\lambda}(x'))\rangle_0\}\,\mathrm{d}^4x\,\mathrm{d}^4x' + \dots \quad (41)$$

Now the expression (34), together with the identities (19) and (20) are used to find the following expressions for the propagators appearing in (41):

$$\langle T(V_{\mu}(x), V_{\nu}(x'))\rangle_0 = -\mathrm{i}d_{\mu\nu}(\partial)\Delta_{\mathrm{c}}(x-x') - \frac{\mathrm{i}}{\mu^2}\eta_{\mu}\eta_{\nu}\delta^{(4)}(x-x')$$

$$\langle T(V_{\mu}(x), \partial'_{\lambda}V_{\rho}(x')-\partial'_{\rho}V_{\lambda}(x'))\rangle_0 = -\mathrm{i}(\partial'_{\lambda}d_{\mu\rho}(\partial)-\partial'_{\rho}d_{\mu\lambda}(\partial))\Delta_{\mathrm{c}}(x-x')$$

$$\langle T(\partial_{\mu}V_{\nu}(x)-\partial_{\nu}V_{\mu}(x), V_{\rho}(x'))\rangle_0 = -\mathrm{i}(\partial_{\mu}d_{\nu\rho}(\partial)-\partial_{\nu}d_{\mu\rho}(\partial))\Delta_{\mathrm{c}}(x-x')$$

$$\langle T(\partial_{\mu}V_{\nu}(x)-\partial_{\nu}V_{\mu}(x), \partial'_{\lambda}V_{\rho}(x')-\partial'_{\rho}V_{\lambda}(x'))\rangle_0 = -\mathrm{i}(\partial_{\mu}\partial'_{\lambda}d_{\nu\rho}(\partial)$$

$$+\partial_{\nu}\partial'_{\rho}d_{\mu\lambda}(\partial)-\partial_{\mu}\partial'_{\rho}d_{\nu\lambda}(\partial)-\partial_{\nu}\partial'_{\lambda}d_{\mu\rho}(\partial))\Delta_{\mathrm{c}}(x-x')$$

$$+\mathrm{i}(g_{\mu\lambda}\eta_{\nu}\eta_{\rho}+g_{\nu\rho}\eta_{\mu}\eta_{\lambda}-g_{\mu\rho}\eta_{\nu}\eta_{\lambda}-g_{\nu\lambda}\eta_{\mu}\eta_{\rho})\delta^{(4)}(x-x').$$

It is clear by inspection that a substitution of these expressions into (41) leads to the desired cancellations, and thus the prescription given by equations (39) and (40) may be used for the calculation of the S matrix.

4. The equivalence theorem

The two theories set out in §§ 2 and 3 are, as they stand, inequivalent. To see this consider them in the forms where the effective Hamiltonians (28) and (39), and the T^* products (29) and (40) are used. In these forms the two theories bear a striking resemblance to one another. For, as may be checked by solving the wave equations in momentum space and applying the normalization condition (Takahashi 1969), the wavefunctions of the pairs of interaction picture fields $\frac{1}{2}(\partial_{\mu}V_{\nu}(x)-\partial_{\nu}V_{\mu}(x))$, $(\mu/\sqrt{2})\phi_{\mu\nu}(x)$ and $V_{\mu}(x)$, $(\sqrt{2}/\mu)\partial^{\nu}\phi_{\mu\nu}(x)$ are in each case exactly the same. Thus also the forms of the interaction Hamiltonians (28) and (31) are effectively the same, and the only place that an inequivalence can arise is in the form of the T^* products for the fields. To see how this leads to inequivalence consider for example the following

$$\langle T^*(\phi_{\mu\nu}(x), \phi_{\lambda\rho}(x'))\rangle_0\frac{\mu^2}{2} = \frac{\mathrm{i}}{2}(A_{\mu\nu\lambda\rho}(\partial)-(\partial^2+\mu^2)I_{\mu\nu\lambda\rho})\Delta_{\mathrm{c}}(x-x')$$

and

$$\langle T^*(\partial_{\mu}V_{\nu}(x)-\partial_{\nu}V_{\mu}(x), \partial'_{\lambda}V_{\rho}(x')-\partial'_{\rho}V_{\lambda}(x'))\rangle_0\frac{1}{4} = \frac{\mathrm{i}}{2}A_{\mu\nu\lambda\rho}(\partial)\Delta_{\mathrm{c}}(x-x')$$

which are just the expressions given respectively at the ends of §§ 2 and 3. Evidently these two expressions just differ by a term proportional to

$$(\partial^2+\mu^2)\Delta_c(x-x') = -\delta^{(4)}(x-x')$$

that is by a contact-type term. Similar remarks hold for the other T^* products, and it is seen that the two theories just differ by the contact interactions necessary to ensure the equality of the corresponding T^* products. More precisely, as will be demonstrated below, the theories given by the Lagrangians (35) and

$$\begin{aligned}\mathscr{L}(x) = {} & \overline{\psi}(x)(\mathrm{i}\not{\partial}-m)\psi(x)+\frac{1}{2}\phi^{\mu\nu}(x)(A_{\mu\nu\lambda\rho}(\partial)+\mu^2 l_{\mu\nu\lambda\rho})\phi^{\lambda\rho}(x)\\ & -\frac{\mu}{\sqrt{2}}\boldsymbol{J}_{\mu\nu}(x)\phi^{\mu\nu}(x)-\frac{1}{\sqrt{2}\,\mu}(\boldsymbol{J}_\mu(x)\partial_\nu-\boldsymbol{J}_\nu(x)\partial_\mu)\phi^{\mu\nu}(x)\\ & +\frac{1}{4}\boldsymbol{J}_{\mu\nu}(x)\boldsymbol{J}^{\mu\nu}(x)-\frac{1}{2\mu^2}\boldsymbol{J}_\mu(x)\boldsymbol{J}^\mu(x)\end{aligned} \tag{42}$$

are equivalent.

The equations of motion corresponding to (42) are

$$\begin{aligned}(\mathrm{i}\not{\partial}-m)\psi(x) = {} & \frac{\mu g_1}{\sqrt{2}}\sigma_{\mu\nu}\psi(x)\phi^{\mu\nu}(x)+\frac{g_2}{\sqrt{2}\,\mu}(\gamma_\mu\psi(x)\partial_\nu\\ & -\gamma_\nu\psi(x)\partial_\mu)\phi^{\mu\nu}(x)-\frac{g_1^2}{2}\sigma_{\mu\nu}\psi(x)\overline{\psi}(x)\sigma^{\mu\nu}\psi(x)\\ & +\frac{g_2^2}{\mu^2}\gamma_\mu\psi(x)\overline{\psi}(x)\gamma^\mu\psi(x)\end{aligned} \tag{43}$$

$$(A_{\mu\nu\lambda\rho}(\partial)+\mu^2 l_{\mu\nu\lambda\rho})\phi^{\lambda\rho}(x) = \frac{\mu}{\sqrt{2}}\boldsymbol{J}_{\mu\nu}(x)+\frac{1}{\sqrt{2}\,\mu}(\partial_\mu\boldsymbol{J}_\nu(x)-\partial_\nu\boldsymbol{J}_\mu(x)) \tag{44}$$

and it is not difficult to see that the essentials of the calculation of § 2 carry through in exactly the same manner for the equations of motion (43) and (44). Consequently the effective interaction Hamiltonian

$$\begin{aligned}\mathscr{H}_{\mathrm{int}}(x) = {} & \frac{\mu}{\sqrt{2}}J_{\mu\nu}(x)\phi^{\mu\nu}(x)+\frac{1}{\sqrt{2}\,\mu}(J_\mu(x)\partial_\nu-J_\nu(x)\partial_\mu)\phi^{\mu\nu}(x)\\ & -\frac{1}{4}J_{\mu\nu}(x)J^{\mu\nu}(x)+\frac{1}{2\mu^2}J_\mu(x)J^\mu(x)\end{aligned} \tag{45}$$

may be used in conjunction with the T^* product to calculate the S matrix corresponding to (42).

To see that the S matrix, so calculated, is the same as the S matrix calculated in terms of (39) and (40), consider the effect of the insertion of the two contact interactions, appearing in (45), in the expression (30). It is not difficult to see that, to second order in the coupling constants, they cause sufficient cancellations for the effective interaction Hamiltonian

$$\mathscr{H}'_{\mathrm{int}}(x) = \frac{\mu}{\sqrt{2}}J_{\mu\nu}(x)\phi^{\mu\nu}(x)+\frac{1}{\sqrt{2}\,\mu}(J_\mu(x)\partial_\nu-J_\nu(x)\partial_\mu)\phi^{\mu\nu}(x) \tag{46}$$

to be used in conjunction with the T^{**} product, defined by

$$\langle T^{**}(\phi_{\mu\nu}(x), \phi_{\lambda\rho}(x'))\rangle_0 = \frac{\mathrm{i}}{\mu^2} A_{\mu\nu\lambda\rho}(\partial)\Delta_c(x-x') \tag{47}$$

$$\langle T^{**}(\partial^\mu \phi_{\mu\nu}(x), \partial'^\lambda \phi_{\lambda\rho}(x'))\rangle_0 = \frac{\mathrm{i}\mu^2}{2} d_{\nu\rho}(\partial)\Delta_c(x-x') \tag{48}$$

etc, to calculate the S matrix. The extension to all orders again follows the method of Kyriakopoulos (1969). In view of equations (46)–(48), and the remarks concerning wavefunctions made at the beginning of this section, it is now evident that the Lagrangians (35) and (42) lead to the same S matrix.

An alternative approach to the proof of equivalence is to find the transformation of field variables which takes the Lagrangians (35) and (42), and their equations of motion (36), (37) and (43), (44) into each other. The required transformation is

$$\boldsymbol{\phi}_{\mu\nu}(x) = \frac{1}{\sqrt{2}\,\mu}(\partial_\mu \boldsymbol{V}_\nu(x) - \partial_\nu \boldsymbol{V}_\mu(x) + \boldsymbol{J}_{\mu\nu}(x)) \tag{49}$$

and

$$\partial^\mu \boldsymbol{\phi}_{\mu\nu}(x) = -\frac{1}{\sqrt{2}\,\mu}(\mu^2 \boldsymbol{V}_\nu(x) - \boldsymbol{J}_\nu(x)). \tag{50}$$

It is evident by inspection that the above transformation takes equations (36) and (43) into each other. To see the connection between equations (37) and (44), (44) is first differentiated by contraction with ∂^μ to give

$$\partial^2 \partial^\rho \boldsymbol{\phi}_{\rho\nu}(x) + \mu^2 \partial^\rho \boldsymbol{\phi}_{\rho\nu}(x) = \frac{\mu}{\sqrt{2}} \partial^\rho \boldsymbol{J}_{\rho\nu}(x) + \frac{1}{\sqrt{2}\,\mu}(\partial^2 \boldsymbol{J}_\nu(x) - \partial_\nu \partial^\rho \boldsymbol{J}_\rho(x)) \tag{51}$$

where the antisymmetry of $\boldsymbol{\phi}_{\mu\nu}(x)$ has been used. Now a use of (50), together with the expression obtained from it by contraction with ∂^ν, namely

$$\mu^2 \partial^\mu \boldsymbol{V}_\mu(x) - \partial^\mu \boldsymbol{J}_\mu(x) = 0$$

reduces (51) to (37). The transformation is reversed by differentiating (37) by ∂_ρ, antisymmetrizing and then using the equations (49) and

$$\partial_\mu \partial^\rho \boldsymbol{\phi}_{\rho\nu}(x) - \partial_\nu \partial^\rho \boldsymbol{\phi}_{\rho\mu}(x) = \frac{1}{\sqrt{2}\,\mu}(\partial^2 \partial_\mu \boldsymbol{V}_\nu(x) - \partial^2 \partial_\nu \boldsymbol{V}_\mu(x) + \partial_\mu \partial^\rho \boldsymbol{J}_{\rho\nu}(x) - \partial_\nu \partial^\rho \boldsymbol{J}_{\rho\mu}(x))$$

which follows from (49) by the appropriate differentiations.

To verify that the transformation (49) and (50) takes the Lagrangians (35) and (42) into one another requires a certain amount of juggling with 4-divergence terms, and so perhaps the simplest way of doing it is through the intermediate stage of the Lagrangian

(42) written wholly in terms of the field $\partial^\mu \phi_{\mu\nu}(x)$, namely

$$\begin{aligned}\mathscr{L}(x) = {} & \bar{\psi}(x)(\mathrm{i}\not{\partial} - m)\psi(x) + \frac{1}{\mu^2}\partial^\rho \phi_{\rho\mu}(x)\{(\partial^2 + \mu^2)g^{\mu\nu} - \partial^\mu \partial^\nu\}\partial^\alpha \phi_{\alpha\nu}(x) \\ & -\frac{\sqrt{2}}{\mu}\partial^\alpha \boldsymbol{J}_{\alpha\mu}(x)\partial_\rho \phi^{\rho\mu}(x) - \frac{\sqrt{2}}{\mu^3}(\partial^2 \boldsymbol{J}_\mu(x) - \partial_\mu \partial_\nu \boldsymbol{J}^\nu(x))\partial_\rho \phi^{\rho\mu}(x) \\ & + \frac{1}{4}\boldsymbol{J}_{\mu\nu}(x)\boldsymbol{J}^{\mu\nu}(x) - \frac{1}{2\mu^2}\boldsymbol{J}_\mu(x)\boldsymbol{J}^\mu(x) \\ & - \frac{1}{4}\left(\boldsymbol{J}_{\mu\nu}(x) + \frac{1}{\mu^2}(\partial_\mu \boldsymbol{J}_\nu(x) - \partial_\nu \boldsymbol{J}_\mu(x))\right)^2 \end{aligned} \tag{52}$$

where $\psi(x)$ and $\partial^\mu \phi_{\mu\nu}(x)$ are to be taken as the basic field variables with respect to which variations will be made, and the field $\phi_{\mu\nu}(x)$ is defined to be connected to $\partial^\mu \phi_{\mu\nu}(x)$ through equation (44). Then a tedious calculation, the details of which are omitted, shows that the Lagrangians (35) and (42) are connected, through the intermediate stage (52), by the transformation defined by (44) and (50).

5. Discussion and conclusions

In § 2 the method of Takahashi and Umezawa (1953) (Takahashi 1969) was used to calculate the interaction Hamiltonian corresponding to the Lagrangian (7), and the generalized Matthews' theorem (Matthews 1949, see also Takahashi 1969 for a more detailed discussion), concerning the neglect of the so called normal-dependent terms, which appear in the propagators and the interaction Hamiltonian, was found to be applicable. This contrasts with certain remarks made by Kyriakopoulos (1969).

Firstly, the remark, made in that paper, that the method of Takahashi and Umezawa is not applicable to the case of the antisymmetric tensor field, seems to stem from an unwillingness to consider the possible appearance of terms proportional to the Klein–Gordon operator in the expression for the Klein–Gordon divisor; whereas, as may be seen in equation (4) of the present paper, such terms are necessary. Secondly, the breakdown of the generalized Matthews' theorem in the example of Kyriakopoulos, allegedly caused by the appearance of contact terms in the effective Hamiltonian (namely that to be used in conjunction with the T^* product), is only apparent. To see this, it should be noted that the sole reason for the appearance of these contact terms is to compensate for the fact that the choice of T^* product, which is made by Kyriakopoulos, is not the correct one, as is given in equation (29) of the present paper. Thus when these contact terms are just considered implicitly, through their effects on the free-particle propagators, the correct form, (29), for the T^* product obtains and the generalized Matthews' theorem is satisfied.

Although the antisymmetric tensor and vector fields are equivalent for the description of free spin one particles, the example studied in the present paper indicates that, on the introduction of interactions, the issue becomes far more complicated. For, as was seen in § 4, an interaction involving the antisymmetric tensor and Dirac fields, and the analogous interaction involving the vector and Dirac fields give rise to inequivalent theories; whilst, since the only *practical* differences between the above two theories arise in the forms of the propagators, equivalent theories may be obtained only at the cost of complicating either interaction by the introduction of contact terms.

This last remark is of importance, since, once it is realized that the essential differences between analogous theories, involving the above spin one fields, arise only from the different forms of the propagators, it becomes evident that, given *any* interaction involving the antisymmetric tensor field the equivalent interaction involving the vector field will only differ from the former by the contact terms necessary to compensate for the differences in the propagators. It follows that, although *a priori* there is an arbitrariness both in the choice of field to describe a spin one particle, and in the choice of interaction, the real choice is not so wide. For given, say, the antisymmetric tensor field and the choice of interaction involving it, a theory can be found which is equivalent to any given theory involving the vector field, and vice versa.

Finally, it is conjectured that this situation will also be true for any spin one field. That is, the set of all theories, arising from the possible choices of interaction involving any given spin one field, includes all the theories arising from the possible choices of interaction involving any spin one field. If true this conjecture would also be expected to generalize to any spin.

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